

Elliptic spectral parameter and infinite dimensional Grassmann variety *

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Abstract

Recent results on the Grassmannian perspective of soliton equations with an elliptic spectral parameter are presented along with a detailed review of the classical case with a rational spectral parameter. The non-linear Schrödinger hierarchy is picked out for illustration of the classical case. This system is formulated as a dynamical system on a Lie group of Laurent series with factorization structure. The factorization structure induces a mapping to an infinite dimensional Grassmann variety. The dynamical system on the Lie group is thereby mapped to a simple dynamical system on a subset of the Grassmann variety. Upon suitable modification, almost the same procedure turns out to work for soliton equations with an elliptic spectral parameter. A clue is the geometry of holomorphic vector bundles over the elliptic curve hidden (or manifest) in the zero-curvature representation.

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1 Introduction

Since the first proposal two decades ago by Sato [20], Segal and Wilson [21], the Grassmannian perspective of soliton equations has been successful for a variety of cases, even including higher dimensional analogues such as the Bogomolny equation and the self-dual Yang-Mills equations [23]. The fundamental observation of this perspective is that a soliton equation can be translated to a simple (essentially linear) dynamical system on a subset of an infinite dimensional

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“universal” Grassmann variety. Almost all of the cases thus examined, however, are equations with a *rational* zero-curvature representation, namely, equations whose zero-curvature equation is made of matrices depending rationally on a spectral parameter. The status of soliton equations related to an elliptic or higher genus algebraic curve has still remained rather obscure, though a few notable studies [4, 2] were done on the Landau-Lifshitz equation (a typical soliton equation with a zero-curvature representation made of elliptic functions [22, 3]).

Recent advances [1, 12, 14] have revealed the existence of a wide class of new integrable PDE’s with a zero-curvature representation constructed on an algebraic curve of arbitrary genus. These equations, too, may be called “soliton equations” in a loose sense, namely, without implying the existence of soliton or soliton-like solutions. The works of Ben-Zvi and Frenkel [1] and Levin, Olshanetsky and Zotov [14] both stem from the notion of the Hitchin systems [7], and aim to obtain an integrable PDE as a $1 + 1$ dimensional analogue of the Hitchin systems. On the other hand, Krichever [12] uses the so called Tyurin parameters to construct Lax or zero-curvature equations on an algebraic curve. The notion of Tyurin parameters originates in algebraic geometry of holomorphic vector bundles over algebraic curves [26], and was applied by Krichever and Novikov in 1970’s to the study of commutative rings of differential operators [9, 10, 11]. Krichever and Levin et al. illustrate their general scheme with several examples related to an elliptic curve. These examples can be used as valuable material for case studies.

One will naturally ask whether these new “soliton equations” can be understood in the Grassmannian perspective. An affirmative answer to this question has been obtained in the simplest case [24, 25], namely, a few examples that have a zero-curvature representation with 2×2 matrices defined on an elliptic curve. Although this is indeed a case study, the upshot clearly shows that a similar result holds in a general and universal form. What distinguishes between the new and conventional soliton equations is the structure of a holomorphic bundle on the relevant algebraic curve. The aforementioned new equations are accompanied by a nontrivial bundle, which plays a central role in both the zero-curvature representation and the Grassmannian perspective. This article presents an outline of these results.

This article is organized as follows. The first half (Sections 2, 3 and 4) of this article is a review on conventional soliton equations with a rational zero-curvature representation. The nonlinear Schrödinger hierarchy is picked out for illustration. This system consists of an infinite number of evolution equations including the nonlinear Schrödinger equation itself in the lowest $1 + 1$ dimensional sector. One can reformulate this system as a dynamical system on a Lie group of Laurent series with factorization structure. The factorization induces a mapping to an infinite dimensional Grassmann variety. The dynamical system on the Lie group is thereby mapped to a simple dynamical system on a subset of the Grassmann variety. This example shows a typical way the usual soliton equations are treated in the Grassmannian perspective. The second half (Section 5, 6 and 7) of this article presents the results on elliptic analogues [24, 25]. Two different types of elliptic analogues are considered here. The

first case is an elliptic analogue of the nonlinear Schrödinger hierarchy. This system is constructed along the line of Krichever's scheme based on Tyurin parameters. The second case is concerned with the Landau-Lifshitz equation and an associated hierarchy of evolution equations. In both cases, a variant of the factorization is formulated as a Riemann-Hilbert problem with respect to the holomorphic bundle structure, and used to define a mapping to an infinite dimensional Grassmann variety.

2 Nonlinear Schrödinger hierarchy

The construction of the nonlinear Schrödinger hierarchy starts from the first order matrix differential operator $\partial_x - A(\lambda)$, where $A(\lambda)$ a 2×2 matrix of the form

$$(1) \quad A(\lambda) = \begin{pmatrix} \lambda & u \\ v & -\lambda \end{pmatrix}.$$

u and v are fields on the one dimensional space, $u = u(x)$, $v = v(x)$, and λ is a rational spectral parameter. From the point of view of affine Lie algebras, it is also natural to express $A(\lambda)$ as

$$(2) \quad A(\lambda) = J\lambda + A^{(1)},$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

Generalities and backgrounds of this kind of soliton equations can be found in Frenkel's lectures [5].

2.1 Generating functions

The first step of the formulation of the hierarchy is to construct a 2×2 matrix of generating functions

$$U(\lambda) = \sum_{n=0}^{\infty} U_n \lambda^{-n}, \quad U_0 = J,$$

that satisfies the differential equation

$$(3) \quad [\partial_x - A(\lambda), U(\lambda)] = 0.$$

This reduces to the differential equations

$$(4) \quad \partial_x U_{n-1} = [J, U_n] + [A^{(1)}, U_{n-1}]$$

for U_n 's. One can, in principle, solve these equations, by a subtle procedure decomposing the equations into the diagonal and off-diagonal part; a similar

procedure is used below to construct another generating function $\phi(\lambda)$. This, however, leaves large arbitrariness in the solution. Moreover, this is by no means an effective way.

These problems are resolved by imposing the algebraic constraint

$$(5) \quad U(\lambda)^2 = I.$$

This amounts to finding $U(\lambda)$ in such a form as

$$(6) \quad U(\lambda) = \phi(\lambda)J\phi(\lambda)^{-1},$$

where $\phi(\lambda)$ is another matrix of generating function

$$\phi(\lambda) = \sum_{n=0}^{\infty} \phi_n \lambda^{-n}, \quad \phi_0 = I,$$

that satisfies the differential equation

$$(7) \quad \partial_x \phi(\lambda) = A(\lambda)\phi(\lambda) - \phi(\lambda)J\lambda.$$

Remarkably, if the *existence* of a solution of this equation is ensured, one can uniquely determine U_n 's by a set of recurrence relations as follows.

Note that the algebraic constraint implies the algebraic relations

$$(8) \quad 0 = JU_n + U_n J + \sum_{m=1}^{n-1} U_m U_{n-m}, \quad n > 0,$$

that hold for U_n 's. Combining this with the differential equations

$$\partial_x U_{n-1} = JU_n - U_n J + [A^{(1)}, U_{n-1}],$$

one obtains the relations

$$(9) \quad 2JU_n = \partial_x U_{n-1} - [A^{(1)}, U_{n-1}] - \sum_{m=1}^{n-1} U_m U_{n-m}.$$

These relations take the form of recurrence relations, which enables one to calculate U_n 's successively. The first few terms read

$$U_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -\frac{1}{2}uv & \frac{1}{2}\partial_x u \\ -\frac{1}{2}\partial_x v & \frac{1}{2}uv \end{pmatrix}, \quad \text{etc.}$$

The matrix elements of U_n thus obtained are “local” quantities, namely, polynomials of derivatives of u, v .

What is left is to prove that the second generating function $\phi(\lambda)$ does exist. The equations for the Laurent coefficients of $\phi(\lambda)$ read

$$(10) \quad \partial_x \phi_n = [J, \phi_{n+1}] + A^{(1)}\phi_n.$$

One can split this matrix equation into the diagonal and off-diagonal part. This results in the two equations

$$\partial_x(\phi_n)_{\text{diag}} = (A^{(1)}\phi_n)_{\text{diag}}$$

and

$$\partial_x(\phi_n)_{\text{off-diag}} = [J, (\phi_{n+1})_{\text{off-diag}}] + (A^{(1)}\phi_n)_{\text{off-diag}}$$

for the diagonal part $(\phi_n)_{\text{diag}}$ and the off-diagonal parts $(\phi_n)_{\text{off-diag}}$ of ϕ_n . The first equation determines the diagonal part of ϕ_n up to integration constants. The second equation is rather an algebraic equation that determines the off-diagonal part of ϕ_{n+1} from the lower coefficients ϕ_1, \dots, ϕ_n . One can thus construct a solution of these equations.

Note that, unlike the aforementioned construction of U_n 's, the construction of $\phi(\lambda)$ is not purely algebraic (the outcome is accordingly “nonlocal”) and leaves large arbitrariness. It is, however, $\phi(\lambda)$ rather than $U(\lambda)$ that plays a more fundamental role in the passage to the Grassmannian perspective.

2.2 Formulation of hierarchy

Let $t = (t_1, t_2, \dots)$ be a sequence of “time” variables; the first one t_1 is to be identified with the spatial variable x . The n -th time evolution is generated by the matrix

$$(11) \quad A_n(\lambda) = \sum_{m=0}^n U_m \lambda^{n-m} = (U(\lambda)\lambda^n)_+.$$

Here $(\cdot)_+$ denotes the polynomial part of a Laurent series of λ . Having introduced these matrices, one can define the nonlinear Schrödinger hierarchy as the system of the Lax equations

$$(12) \quad [\partial_{t_n} - A_n(\lambda), U(\lambda)] = 0$$

for $n = 1, 2, \dots$. Since $A_1(\lambda) = A(\lambda)$, one can identify t_1 with x . In many aspects, this formulation of the nonlinear Schrödinger hierarchy resembles the formulation of the KP hierarchy [20]. For instance, as known in the case of the KP hierarchy, the system of Lax equations is equivalent to the system of zero-curvature equations

$$(13) \quad [\partial_{t_m} - A_m(\lambda), \partial_{t_n} - A_n(\lambda)] = 0$$

for $m, n = 1, 2, \dots$, namely, one can derive one from the other. The zero-curvature equation for $m = 1$ and $n = 2$ gives the equations

$$\partial_t u - \frac{1}{2} \partial_x^2 u + u^2 v = 0, \quad \partial_t v + \frac{1}{2} \partial_x^2 v - uv^2 = 0,$$

which turn into the usual nonlinear Schrödinger equation by rescaling the variables as $u \rightarrow e^{at}u$, $v \rightarrow e^{-at}v$, $t \rightarrow it$ ($a = \pm 1$) and imposing the reality condition $v = \bar{u}$.

Yet another formulation of the hierarchy is achieved by the system of differential equations

$$(14) \quad \partial_{t_n} \phi(\lambda) = A_n(\lambda) \phi(\lambda) - \phi(\lambda) J \lambda^n.$$

If one introduces the so called “formal Baker-Akhiezer function”

$$(15) \quad \psi(\lambda) = \phi(\lambda) \exp\left(\sum_{n=1}^{\infty} t_n J \lambda^n\right),$$

the foregoing equations turn into the auxiliary linear equations

$$(16) \quad \partial_{t_n} \psi(\lambda) = A_n(\lambda) \psi(\lambda).$$

The Frobenius integrability condition of these equations yields the aforementioned zero-curvature equations. On the other hand, if one rewrites the definition of $A_n(\lambda)$ as

$$(17) \quad A_n(\lambda) = \left(\phi(\lambda) J \lambda^n \phi(\lambda)^{-1}\right)_+$$

and insert it into the differential equations for $\phi(\lambda)$, the outcome is a system of nonlinear evolution equations for $\phi(\lambda)$ of the form

$$(18) \quad \partial_{t_n} \phi(\lambda) = -\left(\phi(\lambda) J \lambda^n \phi(\lambda)^{-1}\right)_- \phi(\lambda),$$

where $(\cdot)_-$ denotes the negative power part of a Laurent series of λ . These equations may be thought of as the most fundamental because the Lax and zero-curvature equations can be derived from these equations.

3 Nonlinear Schrödinger hierarchy as dynamical system on Lie group of Laurent series

The nonlinear Schrödinger hierarchy can be interpreted as a dynamical system on an infinite dimensional Lie group. It is customary to formulate such a statement in terms of a loop group, namely, the set of a suitable class of (smooth, real-analytic or square-integrable) mappings from $S^1 = \{\lambda \in \mathbf{C} \mid |\lambda^{-1}| = a\}$ to $\mathrm{SL}(2, \mathbf{C})$. From an aesthetic point of view, however, fixing a circle is not beautiful; the circle is a kind of artifact that did not exist in the formulation of the hierarchy itself. A better approach is to use a Lie group of Laurent series that converge in a neighborhood of $\lambda = \infty$ except at the point $\lambda = \infty$.

3.1 Lie algebras and groups of Laurent series

Let \mathfrak{g} denote the Lie algebra of Laurent series of the form

$$(19) \quad X(\lambda) = \sum_{n=-\infty}^{\infty} X_n \lambda^n, \quad X_n \in \mathfrak{sl}(2, \mathbf{C}),$$

that converge in a neighborhood of $\lambda = \infty$ except at $\lambda = \infty$. In other words, the coefficients are assumed to satisfy the conditions

$$\lim_{n \rightarrow \infty} |X_n|^{1/n} = 0, \quad \limsup_{n \rightarrow \infty} |X_{-n}|^{1/n} < \infty.$$

This Lie algebra has the direct sum decomposition

$$(20) \quad \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-,$$

where \mathfrak{g}_{\pm} are subalgebras of the form

$$\begin{aligned} \mathfrak{g}_+ &= \{X(\lambda) \in \mathfrak{g} \mid X_n = 0 \text{ for } n < 0\}, \\ \mathfrak{g}_- &= \{X(\lambda) \in \mathfrak{g} \mid X_n = 0 \text{ for } n \geq 0\}. \end{aligned}$$

The direct sum decomposition induces a factorization of the associated Lie group $G = \exp \mathfrak{g}$ to the subgroups $G_{\pm} = \exp \mathfrak{g}_{\pm}$, namely, any element $g(\lambda)$ of G near the unit matrix I can be uniquely factorized as

$$(21) \quad g(\lambda) = g_+(\lambda)^{-1} g_-(\lambda), \quad g_{\pm}(\lambda) \in G_{\pm}.$$

Analytically, this is nothing but the so called Riemann-Hilbert problem. In geometric terms, $g(\lambda)$ is the transition function of a holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundle P over \mathbf{P}^1 obtained by gluing trivial bundles over two disks D_+ and D_- , $D_+ \cup D_- = \mathbf{P}^1$, as

$$(22) \quad P = D_+ \times \mathrm{SL}(2, \mathbf{C}) \sqcup D_- \times \mathrm{SL}(2, \mathbf{C}) / \sim,$$

where $(\lambda, g_+) \in D_+ \times \mathrm{SL}(2, \mathbf{C})$ and $(\lambda, g_-) \in D_- \times \mathrm{SL}(2, \mathbf{C})$ are identified if $g_+ = g(\lambda)g_-$. Factorizability of $g(\lambda)$ amounts to holomorphic triviality of P and of associated vector bundles.

3.2 Factorization method

Let $\phi(\lambda)$ denote an arbitrary element of G_- and consider the factorization problem

$$(23) \quad \begin{aligned} \phi(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right) &= \chi(t, \lambda)^{-1} \phi(t, \lambda), \\ \chi(t, \lambda) &\in G_+, \quad \phi(t, \lambda) \in G_-. \end{aligned}$$

Lemma 1 *If t is sufficiently small, the factorization problem (23) has a unique solution.*

Proof. Since $\phi(t, \lambda)$ is expected to be a small deformation of $\phi(\lambda)$, one can assume it in the form

$$\phi(t, \lambda) = \tilde{\chi}(t, \lambda)\phi(\lambda), \quad \tilde{\chi}(t, \lambda) \in G_-,$$

and convert the problem to the form

$$\phi(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right) \phi(\lambda)^{-1} = \chi(t, \lambda)^{-1} \tilde{\chi}(t, \lambda).$$

If t is sufficiently small, the left hand side is close to the unit matrix I , so that one can resort to the local factorizability of G . \square

Suppose that the factorization problem (23) does have a unique solution. The second factor $\phi(t, \lambda)$ then turns out to give a solution of (18):

Theorem 1 *The second factor $\phi(t, \lambda)$ of the factorization problem satisfies the evolution equations (18) and the initial condition $\phi(0, \lambda) = \phi(\lambda)$.*

Proof. If one rewrites the factorization relation as

$$\chi(t, \lambda)\phi(\lambda) = \phi(t, \lambda) \exp\left(\sum_{n=1}^{\infty} t_n J \lambda^n\right)$$

and differentiate both hand sides by t_n , the outcome reads

$$\begin{aligned} & \partial_{t_n} \chi(t, \lambda) \cdot \phi(\lambda) \\ &= \partial_{t_n} \phi(t, \lambda) \cdot \exp\left(\sum_{n=1}^{\infty} t_n J \lambda^n\right) + \phi(t, \lambda) J \lambda^n \exp\left(\sum_{n=1}^{\infty} t_n J \lambda^n\right). \end{aligned}$$

One can use the previous relation once again to eliminate $\phi(\lambda)$ and the exponential from this relation. This yields the relation

$$\partial_{t_n} \chi(t, \lambda) \cdot \chi(t, \lambda)^{-1} = \partial_{t_n} \phi(t, \lambda) \cdot \phi(t, \lambda)^{-1} + \phi(t, \lambda) J \lambda^n \phi(t, \lambda)^{-1}.$$

Let $A_n(t, \lambda)$ denote the 2×2 matrix defined by both hand sides of the last equation. This leads to the two expressions

$$A_n(t, \lambda) = \partial_{t_n} \chi(t, \lambda) \cdot \chi(t, \lambda)^{-1}$$

and

$$A_n(t, \lambda) = \partial_{t_n} \phi(t, \lambda) \cdot \phi(t, \lambda)^{-1} + \phi(t, \lambda) J \lambda^n \phi(t, \lambda)^{-1}$$

of $A_n(t, \lambda)$. The first expression shows that $A_n(t, \lambda)$ takes values in \mathfrak{g}_+ , so that one can replace the right hand side of the second expression by its projection

onto \mathfrak{g}_+ . Since the first term $\partial_{t_n} \phi(t, \lambda) \cdot \phi(t, \lambda)^{-1}$ obviously disappears upon projection, one finds that

$$A_n(\lambda) = \left(\phi(t, \lambda) J \lambda^n \phi(t, \lambda)^{-1} \right)_+.$$

These results show that $\phi(t, \lambda)$ does satisfy (18) as expected. Uniqueness of the factorization implies that $\phi(0, \lambda) = \phi(\lambda)$. \square

This result can be restated in geometric terms as follows. Evolution equations (18) define a dynamical system on G_- . Factorizability of G implies that

$$(24) \quad \begin{array}{ccc} G_- & \rightarrow & G_+ \backslash G \\ \phi(\lambda) & \mapsto & G_+ \phi(\lambda) \end{array}$$

is an injective mapping with open (and dense) image. The dynamical system on G_- is nothing but the pullback, by (24), of the exponential flows

$$(25) \quad G_- g(\lambda) \mapsto G_- g(\lambda) \exp \left(- \sum_{n=1}^{\infty} t_n J \lambda^n \right)$$

on the coset $G_+ \backslash G$. Moreover, the coset $G_+ \backslash G$ may be interpreted as the moduli space of holomorphic $\mathrm{SL}(2, \mathbf{C})$ bundles over \mathbf{P}^1 equipped with trivialization over D_- , elements of the form $G_+ \phi(\lambda)$ being a representative of trivial bundles.

4 Nonlinear Schrödinger hierarchy as dynamical system on infinite dimensional Grassmann variety

The forgoing dynamical system on G_- can be mapped to a dynamical system in an infinite dimensional Grassmann variety. In the literature, two different models of Grassmann varieties (or Grassmann manifolds) have been used for this kind of description. One is Sato's algebraic or complex analytic model based on a vector space of Laurent series [20]; the other is Segal and Wilson's functional analytic model made from the Hilbert space of square-integrable functions on a circle [21]. One should obviously choose Sato's model in the present setting.

4.1 Formulation of Grassmann variety

The Grassmann variety Gr to be used below is constructed from the vector space V of 2×2 matrices $X(\lambda)$ of Laurent series of the form

$$(26) \quad X(\lambda) = \sum_{n=-\infty}^{\infty} X_n \lambda^n, \quad X_n \in \mathrm{gl}(2, \mathbf{C}),$$

that converge in a neighborhood of $\lambda = \infty$ except at the point $\lambda = \infty$. This is almost the same thing as \mathfrak{g} but the coefficients X_n are now an arbitrary matrix;

recall that $\mathfrak{gl}(2, \mathbf{C})$ denotes the vector space (or matrix Lie algebra) of arbitrary 2×2 complex matrices. This vector space is a matrix version of the vector space $V^{\text{ana}(\infty)}$ in Sato's list of models [20]. As noted therein, this vector space has a natural linear topology. The Grassmann variety Gr consists of closed vector subspaces $W \subset V$ satisfying an additional condition as follows:

$$(27) \quad \text{Gr} = \{W \subset V \mid \dim \text{Ker}(W \rightarrow V/V_-) = \dim \text{Coker}(W \rightarrow V/V_-) < \infty\}.$$

Here V_- denotes the vector subspace of V consisting of $X(\lambda)$'s that contain only negative powers of λ :

$$(28) \quad V_- = \{X(\lambda) \in V \mid X_n = 0 \text{ for } n \geq 0\}.$$

The map $W \rightarrow V/V_-$ is the composition of the inclusion $W \hookrightarrow V$ and the canonical projection $V \rightarrow V/V_-$. The so called “big cell” of Gr consists of subspaces $W \subset V$ for which this linear map is an isomorphism:

$$(29) \quad \text{Gr}^\circ = \{W \in \text{Gr} \mid W \simeq V/V_-\}.$$

This is an open subset of Gr , namely, sufficiently small deformations of any element of Gr° remains in Gr° .

4.2 Vacuum and dressing

Let W_0 be the subspace of V spanned by nonnegative powers of λ :

$$(30) \quad W_0 = \{X(\lambda) \in V \mid X_n = 0 \text{ for } n < 0\}.$$

The linear map $W_0 \rightarrow V/V_-$ is obviously isomorphic in view of the basis $\{E_{ij}\lambda^n \mid n \geq 0, i, j = 1, 2\}$ for both vector spaces (E_{ij} are the standard basis of $\mathfrak{gl}(2, \mathbf{C})$). Hence W_0 is an element of the big cell Gr° . This special element of the big cell plays the role of “vacuum,” which corresponds to the vacuum solution $u = v = 0$ of the nonlinear Schrödinger equation.

One can “dress” W_0 by an arbitrary element of G_- :

$$(31) \quad W = W_0\phi(\lambda), \quad \phi(\lambda) = I + \sum_{n=1}^{\infty} \phi_n \lambda^{-n} \in G_-.$$

Lemma 2 *W is an element of the big cell Gr° .*

Proof. W is spanned by $E_{ij}\lambda^n\phi(\lambda)$, $n \geq 0$, $i, j = 1, 2$. By a triangular linear transformation, one can modify this basis of W to another basis $\{w_{n,ij}(\lambda) \mid n \geq 0, i, j = 1, 2\}$ such that

$$w_{n,ij}(\lambda) = E_{ij}\lambda^n + O(\lambda^{-1}).$$

More explicitly,

$$w_{n,ij}(\lambda) = \left(\phi(\lambda) E_{ij} \lambda^n \phi(\lambda)^{-1} \right)_+ \phi(\lambda).$$

The linear map $W \rightarrow V/V_-$ sends this basis to the standard basis $\{E_{ij}\lambda^n \mid n \geq 0, i, j = 1, 2\}$ of V/V_- , thereby turns out to be an isomorphism. \square

The phase space G_- of the dynamical system of the last section can be thus mapped, by the correspondence

$$(22) \quad \phi(\lambda) \mapsto W = W_0\phi(\lambda),$$

to the set

$$(33) \quad \mathcal{M} = \{W \in \text{Gr}^\circ \mid W = W_0\phi(\lambda), \phi(\lambda) \in G_-\}$$

of these “dressed vacua” in (the big cell of) the infinite dimensional Grassmann variety Gr . The problem to be addressed next is to describe the dynamical motion on this new phase space.

Actually, the foregoing mapping $G_- \xrightarrow{\sim} \mathcal{M}$ can be understood in a slightly more general form. Namely, the mapping can be extended to

$$(34) \quad \begin{array}{ccc} G_+ \backslash G & \rightarrow & \text{Gr} \\ G_+ g(\lambda) & \mapsto & W_0 g(\lambda) \end{array}$$

that sends the coset $G_+ \backslash G$ into Gr . Note that this mapping is well defined and injective because

$$g(\lambda) \in G_+ \iff W_0 g(\lambda) = W_0$$

(cf. Lemma 3). Thus, combined with the open embedding (24) of G_- into $G_+ \backslash G$, the mapping $G_- \xrightarrow{\sim} \mathcal{M}$ is substantially the well known embedding of the “affine Grassmannian” $G_+ \backslash G$ into the Sato Grassmannian [21].

4.3 Dynamical system on space of dressed vacua

For simplicity, the following consideration is limited to small values of t . The factorization problem (23) is thereby ensured to have a unique solution. The goal is to elucidate the motion of $W(t) = W_0\phi(t, \lambda) \in \mathcal{M}$. A clue to the answer is the following.

Lemma 3 $W_0\chi(t, \lambda) = W_0$.

Proof. W_0 is obviously closed under multiplication of two element. By construction, $\chi(t, \lambda)$ is obviously an element of W_0 . Therefore $W_0\chi(t, \lambda) \subseteq W_0$. On the other hand, the inverse $\chi(t, \lambda)^{-1}$ is also an element of G_+ as far as t is sufficiently small, so that the same reasoning leads to the conclusion that $W_0\chi(t, \lambda) \subseteq W_0$. Thus the equality follows. \square

If one rewrites the factorization relation (23) as

$$\phi(t, \lambda) = \chi(t, \lambda) \phi(0, \lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right)$$

and insert it into the definition $W(t) = W_0 \phi(t, \lambda)$ of $W(t)$, one finds that

$$\begin{aligned} W(t) &= W_0 \chi(t, \lambda) \phi(0, \lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right) \\ &= W_0 \phi(0, \lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right) \\ &= W(0) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right). \end{aligned}$$

Note that the lemma has been used in the first stage; the first factor $\chi(t, z)$ of the factorization pair is absorbed by W_0 . Thus the motion of the point $W(t)$ of \mathcal{M} turns out to obey the simple exponential law

$$(35) \quad W(t) = W(0) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right).$$

One thus arrives at the following fundamental picture, which is an example of the Grassmannian perspective of soliton equations due to Sato [20] and Segal and Wilson [21].

Theorem 2 *The nonlinear Schrödinger hierarchy can be mapped, by the correspondence $W(t) = W_0 \phi(t, \lambda)$, to a dynamical system on the set \mathcal{M} of dressed vacua in the Grassmann variety Gr . The motion of $W(t)$ obeys the exponential law (35).*

Conversely, given an arbitrary element $\phi(\lambda)$ of G_- , one can derive a solution of the factorization problem (23) from this dynamical system. By Lemma 2, $W(0) = W_0 \phi(\lambda)$ is an element of the big cell. If t is sufficiently small, the point $W(t)$ on the trajectory of the exponential flows (35) still remains in the big cell, because the big cell is an open subset of Gr . This means that the linear map $W(t) \rightarrow V/V_-$, i.e., the composition of the inclusion $W(t) \hookrightarrow V$ and the canonical projection $V \rightarrow V/V_-$, is an isomorphism. Let $\phi(t, \lambda) \in W(t)$ be the inverse image of $I \in V/V_-$ by this isomorphism. Being equal to I modulo V_- , $\phi(t, \lambda)$ is a Laurent series of the form

$$\phi(t, \lambda) = I + \sum_{n=1}^{\infty} \phi_n(t) \lambda^{-n}, \quad \phi_n(t) \in \mathfrak{gl}(2, \mathbf{C}).$$

On the other hand, as an element of

$$W(t) = W_0 \phi(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right),$$

$\phi(t, \lambda)$ can also be expressed as

$$\phi(t, \lambda) = \chi(t, \lambda) \phi(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right)$$

with an element $\chi(t, \lambda)$ of W_0 . Taking the determinant of both hand sides of the last equality yields the equality

$$\det \phi(t, \lambda) = \det \chi(t, \lambda),$$

in which $\phi(\lambda)$ and the exponential disappear because they are known to be unimodular. Notice here that

$$\begin{aligned} \det \phi(t, \lambda) &= 1 + (\text{negative powers of } \lambda), \\ \det \chi(t, \lambda) &= (\text{nonnegative powers of } \lambda). \end{aligned}$$

Consequently, both hand sides of the determinant equality is actually equal to 1. This implies that $\phi(t, \lambda) \in G_-$ and $\chi(t, \lambda) \in G_+$, so that they give a solution of the factorization problem (23).

This shows another aspect of the Grassmannian perspective. Namely, the Grassmann variety can be used as a tool for solving a factorization or Rimann-Hilbert problem. This point of view turns out to be useful later.

5 Elliptic analogue of nonlinear Schrödinger hierarchy

We now turn to examples with an elliptic spectral parameter. The first example is based on an example of Krichever's general construction [12]. Let us briefly recall the background of Krichever's work.

It is well known, after the work of Zakharov and Mikhailov [27], that a naive attempt at the construction of a zero-curvature equation

$$[\partial_x - A(P), \partial_t - B(P)] = 0, \quad P \in \Gamma,$$

on an arbitrary algebraic curve Γ is confronted with a serious difficulty that stems from the Riemann-Roch theorem. If the construction for $\Gamma = \mathbf{P}^1$ also works in the general case, $A(P)$ and $B(P)$ are a matrix of meromorphic functions on Γ with fixed poles, say, Q_1, \dots, Q_s of order m_1, \dots, m_s for $A(P)$ and n_1, \dots, n_s for $B(P)$. Choosing a suitable linearly independent set of meromorphic functions $f_j(P)$, $h = 1, \dots, M$, and $g_k(P)$, $k = 1, \dots, N$, one can expand $A(P)$ and $B(P)$ as

$$A(P) = \sum_{j=1}^M A_j f_j(P), \quad B(P) = \sum_{k=1}^N B_k g_k(P).$$

The (matrix-valued) coefficients A_j, B_k are interpreted as the field variables $A_j = A_j(x, t)$, $B_k = B_k(x, t)$, for which the zero-curvature equation induces a

set of PDE's. Part of these field variables can be eliminated by gauge transformations $A_j \rightarrow g^{-1}A_jg - g_xg^{-1}$, $B_k \rightarrow g^{-1}B_kg - g_tg^{-1}$. In the case where $\Gamma = \mathbf{P}^1$, suitable gauge fixing leads to a determined system of PDE's (i.e., a system of evolution equations) for the reduced field variables. In contrast, if the genus of Γ is not zero, the Riemann-Roch theorem implies that the zero-curvature equation in a "general position" is an overdetermined system for A_j 's and B_k 's. This means that one has to assume some special structure in $A(P)$ and $B(P)$ to obtain a consistent system of evolution equations. An example is the Landau-Lifshitz equation (for which Γ is an elliptic curve).

Krichever [12] pointed out that this difficulty can be avoided by allowing $A(P), B(P)$ to have extra "movable" poles γ_s at which the solutions of the auxiliary linear system $\partial_x\psi(P) = A(P)\psi(P)$, $\partial_t\psi(P) = B(P)\psi(P)$ remain regular. This is reminiscent of the notion of "apparent singularities" in the theory of ordinary differential equations. The number of necessary movable poles turns out to be equal to rg , where r is the size of the matrices $A(P), B(P)$ and g the genus of Γ . Moreover, to each movable pole is assigned a directional vector $\alpha_s \in \mathbf{P}^{r-1}$ as an extra parameter. These pairs (γ_s, α_s) , $s = 1, \dots, rg$, are called "Tyurin parameters" and now join the game as new dynamical variables. The elliptic analogue of the nonlinear Schrödinger hierarchy amounts to the case where $g = 1$ and $r = 2$.

5.1 Matrix of elliptic functions parametrized by Tyurin parameters

The first stage of construction is to choose a suitable counterpart $A(z)$ of $A(\lambda)$. This is a 2×2 matrix of elliptic functions on a nonsingular elliptic curve. The spectral parameter z is now understood to be the standard complex coordinate on the torus $\Gamma = \mathbf{C}/(2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z})$ that realizes the elliptic curve.

In addition to a pole at $z = 0$ (which corresponds to $\lambda = \infty$ in the case of the nonlinear Schrödinger hierarchy), this matrix has two extra poles γ_1, γ_2 , $\gamma_1 \neq \gamma_2$, that depend on x and t_n 's. Two directional vectors $\alpha_1, \alpha_2 \in \mathbf{P}^1$ are introduced as the other half of the Tyurin parameters. These directional vectors can be normalized as $\alpha_s = {}^t(\alpha_s, 1)$. The two fields u, v in the usual nonlinear Schrödinger hierarchy also appear here. Thus one has altogether six dynamical variables $u, v, \gamma_1, \gamma_2, \alpha_1, \alpha_2$ in the formulation of this elliptic analogue.

The matrix $A(z)$ is defined, indirectly, by the following properties:

1. $A(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and is holomorphic at other points.
2. As $z \rightarrow 0$,

$$A(z) = \begin{pmatrix} z^{-1} & u \\ v & -z^{-1} \end{pmatrix} + O(z).$$

3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$A(z) = \frac{\beta_s {}^t\alpha_s}{z - \gamma_s} + O(1),$$

where α_s and β_s are two-dimensional column vectors that do not depend on z . α_s is normalized as $\alpha_s = {}^t(\alpha_s, 1)$.

Lemma 4 *If $\alpha_1 \neq \alpha_2$, a matrix $A(z)$ of meromorphic functions on Γ with these properties does exist. It is unique and can be written explicitly in terms of the Weierstrass zeta function $\zeta(z)$ as*

$$(36) \quad A(z) = \sum_{s=1,2} \beta_s {}^t\alpha_s (\zeta(z - \gamma_s) + \zeta(\gamma_s)) + \begin{pmatrix} \zeta(z) & u \\ v & -\zeta(z) \end{pmatrix},$$

where

$$\beta_1 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} -1 \\ -\alpha_2 \end{pmatrix}, \quad \beta_2 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix}.$$

Proof. One can express $A(z)$ as

$$A(z) = \sum_{s=1,2} \beta_s {}^t\alpha_s \zeta(z - \gamma_s) + J\zeta(z) + C,$$

where C is a constant matrix. By the residue theorem, the coefficients of $\zeta(z - \gamma_1)$, $\zeta(z - \gamma_s)$ and $\zeta(z)$ have to satisfy the linear equation

$$\sum_{s=1,2} \beta_s {}^t\alpha_s + J = 0$$

that ensures that $A(z)$ is single valued on Γ . Solving this equation for β_s leads to the formula stated in the lemma. On the other hand, matching with the Laurent expansion of $A(z)$ at $z = 0$ leads to the relation

$$A^{(1)} = \sum_{s=1,2} \beta_s {}^t\alpha_s \zeta(-\gamma_s) + C,$$

which determines C as shown in the formula above. \square

The final task is to fulfill the requirement on the auxiliary linear system. By Krichever's lemma [12, Lemma 5.2], the auxiliary linear system $\partial_x \psi(z) = A(z)\psi(z)$ has a 2×2 matrix solution that is holomorphic at $z = \gamma_s$ and invertible except at these points if and only if γ_s and α_s satisfy the equations

$$(37) \quad \partial_x \gamma_s + \text{Tr} \beta_s {}^t\alpha_s = 0,$$

$$(38) \quad \partial_x {}^t\alpha_s + {}^t\alpha_s A^{(s,1)} = \kappa_s {}^t\alpha_s,$$

where $A^{(s,1)}$ denotes the constant term of the Laurent expansion of $A(z)$ at $z = \gamma_s$,

$$A^{(s,1)} = \lim_{z \rightarrow \gamma_s} \left(A(z) - \frac{\beta_s {}^t\alpha_s}{z - \gamma_s} \right),$$

and κ_s is a constant to be determined by the equation itself.

5.2 Generating functions

The second stage is to introduce two generating functions

$$\phi(z) = I + \sum_{n=1}^{\infty} \phi_n z^n, \quad U(z) = J + \sum_{n=1}^{\infty} U_n z^n$$

as counterparts of $U(\lambda)$ and $\phi(\lambda)$ in the case of the usual nonlinear Schrödinger hierarchy. Recall that the point $\lambda = \infty$ of \mathbf{P}^1 corresponds to the origin $z = 0$ of the torus Γ .

The first generating function $\phi(z)$ is a Laurent series that satisfies the differential equation

$$(39) \quad \partial_x \phi(z) = A(z) \phi(z) - \phi(z) J z^{-1},$$

where $A(z)$ is understood to be its Laurent expansion

$$(40) \quad A(z) = J z^{-1} + \sum_{n=1}^{\infty} A^{(n)} z^{n-1}$$

at $z = 0$. One can construct a solution of this differential equation by essentially the same (but slightly more complicated) procedure as mentioned in the case of the usual nonlinear Schrödinger hierarchy.

The second generating function $U(z)$ can be obtained from $\phi(z)$ as

$$(41) \quad U(z) = \phi(z) J \phi(z)^{-1},$$

which satisfies the equations

$$(42) \quad [\partial_x - A(z), U(z)] = 0, \quad U(z)^2 = I.$$

The Laurent coefficients are again determined by a set of recurrence relations:

$$(43) \quad 2JU_{n+1} = \partial_x U_n - \sum_{m=1}^{n+1} [A^{(m)}, U_{n+1-m}] - \sum_{m=1}^n U_m U_{n+1-m}.$$

5.3 Construction of hierarchy

The third stage is to construct a set of generators $A_n(z)$, $n = 1, 2, \dots$, of time evolutions. Just like $A(z)$, they are 2×2 matrices of elliptic functions and characterized by the following properties.

1. $A_n(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and is holomorphic at other points.
2. As $z \rightarrow 0$,

$$A_n(z) = U(z) z^{-n} + O(z).$$

3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$A_n(z) = \frac{\beta_{n,s} {}^t\alpha_s}{z - \gamma_s} + O(1),$$

where $\beta_{n,s}$ is a two-dimensional column vector that does not depend on z .

Lemma 5 *If $\alpha_1 \neq \alpha_2$, a matrix $A_n(z)$ of meromorphic functions on Γ with these properties does exist. It is unique and can be written explicitly as*

$$(44) \quad \begin{aligned} A_n(z) = & \sum_{s=1,2} \beta_{n,s} {}^t\alpha_s (\zeta(z - \gamma_s) + \zeta(\gamma_s)) \\ & + \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \partial_z^m \zeta(z) U_{n-1-m} + U_n. \end{aligned}$$

The vectors $\beta_{n,s}$ are determined by the linear equation

$$(45) \quad \sum_{s=1,2} \beta_{n,s} {}^t\alpha_s + U_{n-1} = 0$$

that ensures the single-valuedness of $A_n(z)$ on Γ .

In the following, the genericity condition

$$(46) \quad \alpha_1 \neq \alpha_2$$

is always assumed; the matrices $A(z)$ and $A_n(z)$ are thereby determined. The elliptic analogue of the nonlinear Schrödinger hierarchy is defined by the Lax equations

$$(47) \quad [\partial_{t_n} - A_n(z), U(z)] = 0$$

for the generating function $U(z)$ and the differential equations

$$(48) \quad \partial_{t_n} \gamma_s + \text{Tr} \beta_{n,s} {}^t\alpha_s = 0,$$

$$(49) \quad \partial_{t_n} {}^t\alpha_n + {}^t\alpha_s A_n^{(s,1)} = \kappa_{n,s} \alpha_s$$

for the Tyurin parameters. Here $A_n^{(s,1)}$ denotes the constant term of the Laurent expansion of $A_n(z)$ at $z = \gamma_s$, i.e.,

$$A_n^{(s,1)} = \lim_{z \rightarrow \gamma_s} \left(A_n(z) - \frac{\beta_{n,s} {}^t\alpha_s}{z - \gamma_s} \right),$$

and $\kappa_{n,s}$ is a constant determined by the differential equation itself. (48) and (49) are the necessary and sufficient conditions for the auxiliary linear system $\partial_{t_n} \psi(z) = A_n(z) \psi(z)$ to have a 2×2 matrix solution that is holomorphic at $z = \gamma_s$ and invertible except at these points.

One can confirm that the zero-curvature equations

$$(50) \quad [\partial_{t_m} - A_m(z), \partial_{t_n} - A_n(z)] = 0$$

are satisfied by any solution of the three equations (47), (48) and (49). This implies, in particular, the commutativity of flows generated by $A_n(z)$. Actually, the following stronger statement holds as in the case of the usual nonlinear Schrödinger hierarchy.

Theorem 3 *The system of Lax equations (47) and the system of zero-curvature equations (50) are equivalent under the equations (48) and (49) for the Tyurin parameters.*

As regards the status of (48) and (49), one can derive them from the zero-curvature equations

$$(51) \quad [\partial_{t_n} - A_n(z), \partial_x - A(z)] = 0$$

assuming that (37) and (38) are satisfied. In this respect, (37) and (38) should be understood as part of the definition of $A(z)$. Krichever's construction of a hierarchy is rather based on these zero-curvature equations [12].

6 Elliptic analogue of nonlinear Schrödinger hierarchy in Grassmannian perspective

A technical clue to the Grassmannian perspective of the elliptic analogue of the nonlinear Schrödinger hierarchy is again a factorization or Riemann-Hilbert problem. The situation is, however, far more complicated. First of all, the present case is concerned with the torus rather than the sphere. Moreover, whereas the usual Riemann-Hilbert problem on the sphere is based on the triviality of a holomorphic bundle (cf. Section 3.1), the present case is, by construction, related to a *nontrivial* holomorphic bundle in the Tyurin parameterization. As it turns out, what is relevant to the present setting is a Riemann-Hilbert problem with *degeneration points*; Tyurin parameters are nothing but the geometric data of those points. This kind of Riemann-Hilbert problems also appear in the work of Krichever and Novikov [9, 10, 11] on commutative rings of differential operators.

Another clue can be found in the paper of Previato and Wilson [18]. They demonstrate therein a “dressing method” based on an infinite dimensional Grassmann variety to solve a Riemann-Hilbert problem of the Krichever-Novikov type. Moreover, their paper shows what should be the “vacuum” that corresponds to a holomorphic vector bundle in the Tyurin parametrization.

These ideas lead to a Grassmannian perspective of the elliptic analogue [24].

6.1 Riemann-Hilbert problem with degeneration points

In the following, t denotes the full set of time variables (t_1, t_2, \dots) , in which x is identified with t_1 . Moreover, any quantity that depends on t is written with its t -dependence indicated explicitly as $A(t, z)$, $A_n(t, z)$, $\gamma_s(t)$, $\alpha_s(t)$, etc.

The Lax equations (47) and the zero-curvature equations (50) are associated with the auxiliary linear system

$$\partial_{t_n} \psi(t, z) = A_n(t, z) \psi(t, z).$$

The Riemann-Hilbert problem is concerned with two distinct solutions of this linear system.

One solution is the Laurent series solution of the form

$$(52) \quad \psi(t, z) = \phi(t, z) \exp\left(\sum_{n=1}^{\infty} t_n J z^{-n}\right),$$

where the prefactor $\phi(t, z)$ is a Laurent series of the form

$$\phi(t, z) = I + \sum_{n=1}^{\infty} \phi_n(t) z^n.$$

This prefactor is nothing but the generating function introduced previously, but it is now required to satisfy the differential equations

$$(53) \quad \partial_{t_n} \phi(t, z) = A_n(t, z) \phi(t, z) - \phi(t, z) J z^{-n}$$

for $n = 1, 2, \dots$ as well. Note that this Laurent series solution, by its nature, carries no information on the global structure of $A_n(t, z)$'s.

Another solution $\chi(t, z)$ is characterized by the initial condition

$$(54) \quad \chi(0, z) = I.$$

This solution $\chi(t, z)$ turns out to carry global information. To avoid delicate problems, suppose that the solutions of the hierarchy under consideration are (real or complex) analytic in a neighborhood of the initial point $t = 0$. One can then expand it to a Taylor series in t . The Taylor coefficients of $\chi(t, z)$ at $t = 0$ can be evaluated by successively differentiating the differential equations as

$$\begin{aligned} \partial_{t_n} \chi(t, z) &= A_n(t, z) \chi(t, z), \\ \partial_{t_m} \partial_{t_n} \chi(t, z) &= (\partial_{t_m} A_n(t, z) + A_n(t, z) A_m(t, z)) \chi(t, z), \\ \partial_{t_k} \partial_{t_m} \partial_{t_n} \chi(t, z) &= \left(\partial_{t_k} \partial_{t_m} A_n(t, z) + \partial_{t_k} (A_n(t, z) A_m(t, z)) \right. \\ &\quad \left. + (\partial_{t_m} A_n(t, z)) A_k(t, z) + A_n(t, z) A_m(t, z) A_k(t, z) \right) \chi(t, z), \end{aligned}$$

etc. Letting $t = 0$, we are left with a polynomial of derivatives of A_n 's. One can deduce from these calculations that all Taylor coefficients of $\chi(t, z)$ at $t = 0$ are a

matrix of meromorphic functions of z on Γ with poles at $z = 0, \gamma_1(0), \gamma_2(0)$ and holomorphic at other points. Since the order of poles at $z = 0$ is unbounded for higher orders of the Taylor expansion, the Taylor series of $\chi(t, z)$ has an essential singularity at $z = 0$. On the other hand, the poles at $z = \gamma_1(0), \gamma_2(0)$ remain to be of the first order. More careful analysis [24] shows that the detailed structure of these first order poles:

Lemma 6 *As $z \rightarrow \gamma_s(0)$, $s = 1, 2$, $\chi(t, z)$ behaves as*

$$\chi(t, z) = \frac{\beta_{\chi, s}(t) {}^t\alpha_s(0)}{z - \gamma_s(0)} + O(1),$$

where $\beta_{\chi, s}(t)$ is a two-dimensional vector.

Another important property of $\chi(t, z)$ can be seen from the linear system

$$\partial_x \chi(t, z) = A(t, z) \chi(t, z).$$

Taking the residue at $z = \gamma_s(t)$ yields the relation

$$0 = \beta_s(t) {}^t\alpha_s(t) \chi(t, \gamma_s(t)),$$

which implies that

$${}^t\alpha_s(t) \chi(t, \gamma_s(t)) = \mathbf{0}.$$

Thus one finds the following.

Lemma 7 *$\det \chi(t, z)$ has zeroes at $z = \gamma_s(t)$, $s = 1, 2$. ${}^t\alpha_s(t)$ is a left null vector of $\chi(t, \gamma_s(t))$.*

This result shows that $\chi(t, z)$ is exactly the solution mentioned in Krichever's lemma [12, Lemma 5.2], namely a matrix solution of the auxiliary linear system that is holomorphic at the movable poles of $A(t, z)$.

Since $\psi(t, z)$ and $\chi(t, z)$ satisfy the same auxiliary linear system, their “matrix ratio” $\chi(t, z)^{-1} \psi(t, z)$ is independent of t , hence equal to its initial value at $t = 0$. One thus obtains the relation

$$(55) \quad \psi(0, z) = \chi(t, z)^{-1} \psi(t, z)$$

or, equivalently,

$$(56) \quad \phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right) = \chi(t, z)^{-1} \phi(t, z).$$

This is the Riemann-Hilbert problem that plays the role of an intermediate step towards the Grassmannian perspective. The pair of $\phi(t, z)$ and $\chi(t, z)$ are referred to as a Riemann-Hilbert pair.

Note that this Riemann-Hilbert problem has a few unusual aspects. Firstly, in addition to the pole at $z = 0$, $\chi(t, z)$ has extra poles at $z = \gamma_s(0)$, $s = 1, 2$. Secondly, $\chi(t, z)$ *degenerate* (namely, $\det \chi(t, z)$ has zeroes) at $z = \gamma_s(t)$, $s = 1, 2$. Moreover, these degeneration points are movable as t varies.

6.2 Grassmann variety and vacuum

Let V denote the vector space of all 2×2 matrices of Laurent series

$$(57) \quad X(z) = \sum_{n=-\infty}^{\infty} X_n z^n, \quad X_n \in \mathfrak{gl}(2, \mathbf{C}),$$

that converges in a neighborhood of $z = 0$ except at $z = 0$, and V_+ the subspace

$$(58) \quad V_+ = \{X(z) \in V \mid X_n = 0 \text{ for } n \leq 0\}.$$

of all $X(z) \in V$ that are holomorphic and vanish at $z = 0$. Recalling that z amounts to λ^{-1} , this is essentially the same setting as the case of the nonlinear Schrödinger hierarchy. The Grassmann variety Gr and the big cell $\text{Gr}^\circ \subset \text{Gr}$ are defined as

$$(59) \quad \text{Gr} = \{W \subset V \mid \dim \text{Ker}(W \rightarrow V/V_+) = \dim \text{Coker}(W \rightarrow V/V_+) < \infty\}$$

and

$$(60) \quad \text{Gr}^\circ = \{W \in \text{Gr} \mid W \simeq V/V_+\}.$$

The following lemma shows the construction of a special point $W_0(\gamma, \alpha)$ of Gr° , which plays the role of “vacuum” in the present setting. This is a matrix version of the vacuum that Previato and Wilson [18] suggest to use for a holomorphic vector bundle in the Tyurin parametrization.

Lemma 8 *Let $\gamma = (\gamma_1, \gamma_2)$ be a pair of distinct points of Γ , $\gamma_1 \neq \gamma_2$, and $\alpha = (\alpha_1, \alpha_2)$ a pair of constants satisfying the genericity condition $\alpha_1 \neq \alpha_2$. Then, for any integer $n \geq 0$ and the matrix indices $i, j = 1, 2$, there is a unique 2×2 matrix $w_{n,ij}(z)$ of meromorphic functions on Γ with the following properties:*

1. $w_{n,ij}(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and is holomorphic at other points.
2. $w_{n,ij}(z) = E_{ij} z^{-n} + O(z)$ as $z \rightarrow 0$, where E_{ij} , $i, j = 1, 2$, are the standard basis of $\mathfrak{gl}(2, \mathbf{C})$.
3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$w_{n,ij}(z) = \frac{\beta_{n,ij,s} {}^t \alpha_s}{z - \gamma_s} + O(1),$$

where $\alpha_s = {}^t(\alpha_s, 1)$, and $\beta_{n,ij,s}$ is another two-dimensional constant vector.

The subspace

$$(61) \quad W_0(\gamma, \alpha) = \langle w_{n,ij}(z) \mid n \geq 0, i, j = 1, 2 \rangle$$

spanned by (the Laurent series of) $w_{n,ij}(z)$'s is an element of the big cell.

This vacuum $W_0(\gamma, \alpha)$ is “dressed” by a Laurent series to become a dressed vacuum:

$$W = W_0(\gamma, \alpha)\phi(z), \quad \phi(z) = I + \sum_{n=1}^{\infty} \phi_n z^n, \quad \phi_n \in \mathfrak{gl}(2, \mathbf{C}).$$

The set

$$(62) \quad \mathcal{M} = \{W \in \text{Gr}^o \mid W = W_0(\gamma, \alpha)\phi(z), \phi_n \in \mathfrak{gl}(2, \mathbf{C}), \\ \gamma = (\gamma_1, \gamma_2) \in \Gamma^2, \alpha = (\alpha_1, \alpha_2) \in \mathbf{C}^2, \gamma_1 \neq \gamma_2, \alpha_1 \neq \alpha_2\}$$

of these dressed vacua is the phase space for the Grassmannian perspective of the elliptic analogue of the nonlinear Schrödinger hierarchy.

6.3 Interpretation of Riemann-Hilbert problem

For technical reasons, the following consideration is limited to a small neighborhood of $t = 0$. The goal is to translate the Riemann-Hilbert problem to the language of the set \mathcal{M} of dressed vacua. A clue is the the following.

Lemma 9 $W_0(\gamma(t), \alpha(t))\chi(t, z) = W_0(\alpha(0), \gamma(0))$.

Proof. The following is an outline of the proof; see the paper [24] for details. Let $w_{n,ij}(t, z)$, $n \geq 0$, $i, j = 1, 2$, denote the elements of the basis of $W_0(\gamma(t), \alpha(t))$ defined in Lemma 8. $w_{n,ij}(t, z)$ has poles at $z = 0, \gamma_1(t), \gamma_2(t)$, and behaves as

$$w_{n,ij}(t, z) = \frac{\beta_{n,ij,s}(t) {}^t\alpha_s(t)}{z - \gamma_s(t)} + O(1)$$

as $z \rightarrow \gamma_s(t)$. Upon multiplication with $\chi(t, z)$, the poles at $z = \gamma_s(t)$ are cancelled out because ${}^t\alpha_s(t)$ is a left null vector of $\chi(t, \gamma_s(t))$. Thus $w_{n,ij}(t, z)\chi(t, z)$ turns out to have an essential singularity at $z = 0$, first order poles at $z = \gamma_s(0)$, $s = 1, 2$, and is holomorphic at other points. The leading part of the Laurent expansion at $z = \gamma_s(0)$ takes the form

$$w_{n,ij}(t, z)\chi(t, z) = \frac{w_{n,ij}(t, \gamma_s(0))\beta_{\chi,s}(t) {}^t\alpha_s(0)}{z - \gamma_s(0)} + O(1).$$

These results show that $w_{n,ij}(t, z)\chi(t, z)$ is an element of $W_0(\gamma(0), \alpha(0))$. One can thus see that

$$W_0(\gamma(t), \alpha(t))\chi(t, z) \subseteq W_0(\gamma(0), \alpha(0)).$$

A few more steps of consideration on the analytic properties of $\chi(t, z)$ lead to the conclusion that these two vector subspaces of V are equal. \square

Thanks to this lemma, one can readily convert the Riemann-Hilbert problem to the language of dressed vacua. The Riemann-Hilbert relation yields the relation

$$W_0(\gamma(t), \alpha(t))\phi(t, z) = W_0(\gamma(t), \alpha(t))\chi(t, z)\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right).$$

The lemma shows that $W_0(\gamma(t), \alpha(t))$ absorbs $\chi(t, z)$ to become $W_0(\gamma(0), \alpha(0))$. The outcome is the relation

$$W_0(\gamma(t), \alpha(t))\phi(t, z) = W_0(\gamma(0), \alpha(0))\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right),$$

which means that the dressed vacuum $W(t) = W_0(\gamma(t), \alpha(t))\phi(t, z) \in \mathcal{M}$ obeys the exponential law

$$(63) \quad W(t) = W(0) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right).$$

Conversely, one can derive a solution of the Riemann-Hilbert problem from these exponential flows as follows. (This is a variation of the dressing method of Previato and Wilson [18].) Given a set of initial values $\gamma(0), \alpha(0)$ and $\phi(0, z)$, one can consider the exponential flows sending $W(0) = W_0(\gamma(0), \alpha(0))\phi(0, z)$ to $W(t)$. If t is sufficiently small, $W(t)$ remains in the big cell. This means that the linear map $W(t) \rightarrow V/V_+$ is an isomorphism. Let $\phi(t, z) \in W(t)$ be the inverse image of $I \in V/V_+$ by this isomorphism. Being equal to I modulo V_+ , $\phi(t, z)$ is a Laurent series of the form

$$\phi(t, z) = 1 + \sum_{n=1}^{\infty} \phi_n(t) z^n.$$

On the other hand, as an element of

$$W(t) = W_0(\gamma(0), \alpha(0))\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right),$$

$\phi(t, z)$ can also be expressed as

$$\phi(t, z) = \chi(t, z)\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right)$$

with an element $\chi(t, z)$ of $W_0(\gamma(0), \alpha(0))$. Thus one obtains a Riemann-Hilbert pair. The associated Tyurin parameters $(\gamma_s(t), \alpha_s(t))$ are determined as the position of zeros of $\chi(t, z)$ and the normalized left null vector of $\chi(t, z)$ at those degeneration points.

One thus eventually arrives at the following Grassmannian perspective in the present setting.

Theorem 4 *The elliptic analogue of the nonlinear Schrödinger hierarchy can be mapped, by the correspondence $W(t) = W_0(\gamma(t), \alpha(t))\phi(t, z)$, to a dynamical system on the set \mathcal{M} of dressed vacua in the Grassmann variety Gr . The motion of $W(t)$ obeys the exponential law. Conversely, the exponential flows on \mathcal{M} yield a solution of the Riemann-Hilbert problem.*

As a final remark, it should be stressed that the main characters of this story are all related to the geometry of holomorphic vector bundles over Γ . The Tyurin parameters $(\gamma(t), \alpha(t))$ correspond to a holomorphic vector bundle that deforms as t varies. The subspace $W_0(\gamma, \alpha) \subset V$ can be identified with the space of holomorphic sections of the associated $\text{sl}(2, \mathbf{C})$ bundle over the punctured torus $\Gamma \setminus \{z = 0\}$. $\phi(t, z)$ is related to changing local trivialization of this bundle at $z = 0$. Note, in particular, that the primary role (as a dynamical variable) is now played by the data of local trivialization. This differs decisively from the work of Previato and Wilson [18]; they take, in place of the data of local trivialization, a set of functions in Krichever’s “algebraic spectral data” [9] as main parameters. In this respect, the present setting is rather close to Li and Mulase’s approach [17, 15] to commutative rings of differential operators; they treat the choice of local trivialization as an independent data.

7 Landau-Lifshitz hierarchy in Grassmannian perspective

The last example with an elliptic spectral parameter is the Landau-Lifshitz equation in $1 + 1$ dimensions and the associated hierarchy (Landau-Lifshitz hierarchy) of higher time evolutions. This is one of the classical examples of soliton equations with an elliptic zero-curvature representation [22, 3].

As regards the Grassmannian perspective of this equation, studies from a very close point of view have been done by Date, Jimbo, Kashiwara and Miwa [4] and Carey, Hannabuss, Mason and Singer [2]. Actually, Date et al. developed a free fermion formalism rather than a Grassmannian formalism. Carey et al. presented two approaches to a factorization method for solving the Landau-Lifshitz equation. The first approach uses an infinite dimensional Grassmann manifold (rather than a “variety”, because this is a functional analytic model). The second one is based on the geometry of a holomorphic vector bundle over the torus $\Gamma = \mathbf{C}/(2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z})$. This work is yet unsatisfactory because their usage of the Grassmann manifold fails to incorporate the bundle structure.

The lessons in the preceding examples show that a clue is always the choice of a suitable “vacuum” (and of course a Grassmann variety that accommodates that vacuum). As the paper of Previato and Wilson suggests [18], a correct choice of vacuum is somehow related to the structure of a holomorphic vector bundle. The Grassmannian perspective of the Landau-Lifshitz equation (and hierarchy), too, can be reached along the same lines [25].

7.1 Geometric and algebraic structures behind Landau-Lifshitz equation

The zero-curvature representation of the Landau-Lifshitz equation [22, 3] is based on the first order matrix differential operator $\partial_x - A(z)$ with the A -matrix of the form

$$(64) \quad A(z) = \sum_{a=1,2,3} w_a(z) S_a \sigma_a,$$

where S_a 's are dynamical variables (spin fields) and σ_a 's denote the Pauli matrices. The weight functions $w_a(z)$ are defined by Jacobi's elliptic functions $\text{sn}, \text{cn}, \text{dn}$ as

$$(65) \quad w_1(z) = \frac{\alpha \text{cn}(\alpha z)}{\text{sn}(\alpha z)}, \quad w_2(z) = \frac{\alpha \text{dn}(\alpha z)}{\text{sn}(\alpha z)}, \quad w_3(z) = \frac{\alpha}{\text{sn}(\alpha z)},$$

where $\alpha = \sqrt{e_1 - e_3}$, $e_a = \wp(\omega_a)$.

The matrix $A(z)$ has the twisted double periodicity

$$(66) \quad A(z + 2\omega_a) = \sigma_a A(z) \sigma_a, \quad a = 1, 2, 3,$$

where ω_2 denotes the third half period $\omega_2 = -\omega_1 - \omega_3$. This is a manifestation of the structure of a nontrivial holomorphic $\text{sl}(2, \mathbf{C})$ bundle over the torus; $A(z)$ is a meromorphic section of that bundle. The same bundle is known to play a fundamental role in the elliptic Gaudin model and an associated conformal field theory [13].

Compared with the equations formulated by Tyurin parameters, the Landau-Lifshitz equation is rather close to classical soliton equations with a rational zero-curvature representation, because one can treat this system by a factorization method based on a Lie group of Laurent series (or a loop group) with factorization structure [19, 2]. Geometrically, this fact is related to *rigidity* of the aforementioned holomorphic $\text{sl}(2, \mathbf{C})$ bundle or of an associated $\text{SL}(2, \mathbf{C})$ bundle [8].

To formulate the factorization structure, one starts from a Lie algebra with direct sum decomposition to two subalgebras. Let \mathfrak{g} be the Lie algebra of Laurent series

$$(67) \quad X(z) = \sum_{n=-\infty}^{\infty} X_n z^n, \quad X_n \in \text{sl}(2, \mathbf{C}),$$

that converge in a neighborhood of $z = 0$ except at $z = 0$. This Lie algebra has a direct sum decomposition of the form

$$(68) \quad \mathfrak{g} = \mathfrak{g}_{\text{out}} \oplus \mathfrak{g}_{\text{in}},$$

where \mathfrak{g}_{in} and $\mathfrak{g}_{\text{out}}$ are the following subalgebras:

1. \mathfrak{g}_{in} consists of all $X(z) \in \mathfrak{g}$ that are also holomorphic at $z = 0$, i.e. $X_n = 0$ for $n < 0$.

2. $\mathfrak{g}_{\text{out}}$ consists of all $X(z) \in \mathfrak{g}$ that can be extended to a holomorphic mapping $X : \mathbf{C} \setminus (2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z}) \rightarrow \mathfrak{sl}(2, \mathbf{C})$ with singularity at each point of $2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z}$ and satisfy the twisted double periodicity condition

$$X(z + 2\omega_a) = \sigma_a X(z) \sigma_a \quad a = 1, 2, 3.$$

Note that constant matrices are excluded from $\mathfrak{g}_{\text{out}}$, so that $\mathfrak{g}_{\text{out}} \cap \mathfrak{g}_{\text{in}} = \{0\}$. One can choose $\{\partial_z^n w_a(z) \sigma_a \mid n \geq 0, a = 1, 2, 3\}$ as a basis of $\mathfrak{g}_{\text{out}}$; the projection $(\cdot)_{\text{out}} : \mathfrak{g} \rightarrow \mathfrak{g}_{\text{out}}$ thereby takes the simple form

$$(69) \quad (z^{-n-1} \sigma_a)_{\text{out}} = \frac{(-1)^n}{n!} \partial_z^n w_a(z) \sigma_a, \quad (z^n \sigma_a)_{\text{out}} = 0, \quad n \geq 0.$$

The direct sum decomposition of the Lie algebra \mathfrak{g} induces the factorization of the associated Lie group $G = \exp \mathfrak{g}$ to the subgroups $G_{\text{out}} = \exp \mathfrak{g}_{\text{out}}$ and $G_{\text{in}} = \exp \mathfrak{g}_{\text{in}}$, namely, any element $g(z)$ of G near the unit matrix I can be uniquely factorized as

$$(70) \quad g(z) = g_{\text{out}}(z)^{-1} g_{\text{in}}(z), \quad g_{\text{out}}(z) \in G_{\text{out}}, \quad g_{\text{in}}(z) \in G_{\text{in}}.$$

7.2 Construction of hierarchy

The Landau-Lifshitz hierarchy can be obtained by the projection of the exponential flows

$$(71) \quad g(\lambda) \mapsto g(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n J \lambda^n\right)$$

on G to G_{in} with regard to the foregoing factorization [6, 2]. The fundamental dynamical variable is thus a Laurent series of the form

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n, \quad \det \phi(z) = 1,$$

that converges in a neighborhood of $z = 0$. The time evolution $\phi(0, z) \mapsto \phi(t, z)$ is achieved by the factorization

$$(72) \quad \phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n z^{-n} \sigma_3\right) = \chi(t, z)^{-1} \phi(t, z),$$

where $\chi(t, z)$ is an element of G_{out} that also depends on t . As demonstrated in the case of the usual nonlinear Schrödinger hierarchy, one can derive the equations

$$(73) \quad \partial_{t_n} \phi(t, z) = A_n(t, z) \phi(t, z) - \phi(t, z) z^{-n} \sigma_3,$$

where

$$(74) \quad A_n(t, z) = \left(\phi(t, z) z^{-n} \sigma_3 \phi(t, z)^{-1} \right)_{\text{out}},$$

or

$$(75) \quad \partial_{t_n} \phi(t, z) = - \left(\phi(t, z) z^{-n} \sigma_3 \phi(t, z)^{-1} \right)_{\text{in}} \phi(t, z)$$

as equations of motion of $\phi(t, z) \in G_{\text{in}}$. $(\cdot)_{\text{in}}$ denotes the projection $\mathfrak{g} \rightarrow \mathfrak{g}_{\text{in}}$.

The zero-curvature equations

$$(76) \quad [\partial_{t_m} - A_m(t, z), \partial_{t_n} - A_n(t, z)] = 0$$

follow from the auxiliary linear system

$$(77) \quad (\partial_{t_n} - A_n(t, z))\chi(t, z) = 0$$

as the Frobenius integrability condition.

7.3 Grassmann variety and vacuum

It will be reasonable to use the same pair (V, V_+) of vector spaces as those for the elliptic analogues of the nonlinear Schrödinger hierarchy. Actually, already at this stage, the present approach differs from that of Carey et al. [2]. Carey et al. use a vector space of two-component vectors rather than 2×2 matrices; this is not suited for treating the aforementioned $\mathfrak{sl}(2, \mathbf{C})$ bundle structure.

The next problem is the choice of a suitable subspace $W_0 \subset V$ that plays the role of “vacuum.” In view of the previous examples, W_0 should be a vector subspace that absorbs the first factor $\chi(t, z)$ of the factorization pair. This will be the case if W_0 consists of matrix-valued functions of z with the same analytic properties as $\chi(t, z)$. As a t -dependent element of G_{out} , $\chi(t, z)$ is a matrix-valued holomorphic function on $\mathbf{C} \setminus (2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z})$ with twisted double periodicity.

For this reason, let W_0 be the subspace of V that consists of all $X(z) \in V$ with the following properties:

1. $X(z)$ can be extended to a holomorphic mapping

$$X : \mathbf{C} \setminus (2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z}) \rightarrow \mathfrak{gl}(2, \mathbf{C}).$$

2. $X(z)$ has the twisted double periodicity

$$X(z + 2\omega_a) = \sigma_a X(z) \sigma_a, \quad a = 1, 2, 3.$$

This resembles the definition of $\mathfrak{g}_{\text{out}}$; the difference is, firstly, that $X(z)$ now takes values in $\mathfrak{gl}(2, \mathbf{C})$ rather than $\mathfrak{sl}(2, \mathbf{C})$, and secondly, that $X(z)$ can be a constant matrix.

As it turns out, this subspace W_0 does *not* satisfy the condition in the definition of the Grassmann variety Gr that has been used in the previous case:

Lemma 10 *The following hold for the linear map $W_0 \rightarrow V/V_+$:*

1. $\text{Im}(W_0 \rightarrow V/V_+) \oplus \mathfrak{sl}(2, \mathbf{C}) \oplus \mathbf{C}z^{-1}I = V/V_+$.
2. $\text{Ker}(W_0 \rightarrow V/V_+) = \{0\}$.

Proof. It is a (slightly advanced) exercise of linear algebra and complex function theory to confirm that W_0 is spanned by I , $\partial_z^n w_a(z)\sigma_a$, $a = 1, 2, 3$, and $\partial_z^n \wp(z)I$ for $n \geq 0$. $\partial_z^n w_a(z)$ and $\partial_z^n \wp(z)$ have the Laurent expansion

$$\partial_z^n w_a(z)\sigma_a = (-1)^n n! z^{-n-1} \sigma_a + O(z)$$

and

$$\partial_z^n \wp(z)I = (-1)^n (n+1)! z^{-n-2} I + O(z)$$

at $z = 0$. This implies that these generators of W_0 are linearly independent, and that the image of $W \rightarrow V/V_+$ are spanned by I , $z^{-n-1}\sigma_a$, $a = 1, 2, 3$, and $z^{-n-2}I$ for $n \geq 0$ among the standard basis $\{z^{-n}\sigma_a, z^{-n}I \mid n \geq 0, a = 1, 2, 3\}$ of V/V_+ . What is missing are σ_a , $a = 1, 2, 3$, and $z^{-1}I$, which respectively span the subspaces $\mathfrak{sl}(2, \mathbf{C})$ and $\mathbf{C}z^{-1}$ of V/V_+ . Thus the assertion on $\text{Im}(W_0 \rightarrow V/V_+)$ follows. On the other hand, one has $\text{Ker}(W_0 \rightarrow V/V_+) = W_0 \cap V_+$. Any element $X(z)$ of $W_0 \cap V_+$ has the twisted double periodicity and a zero at all points of $2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z}$; by Liouville's theorem, such a matrix-valued function is identically zero. \square

This lemma implies that

$$(78) \quad \dim \text{Ker}(W_0 \rightarrow V/V_+) = 0, \quad \dim \text{Coker}(W_0 \rightarrow V/V_+) = 4.$$

Consequently, the Grassmann variety to accommodate W_0 is not Gr but the following one:

$$(79) \quad \text{Gr}_{-4} = \{W \subset V \mid \dim \text{Ker}(W \rightarrow V/V_+) = \dim \text{Coker}(W \rightarrow V/V_+) - 4 < \infty\}.$$

The subset

$$(80) \quad \text{Gr}_{-4}^\circ = \{W \in \text{Gr}_{-4} \mid W \simeq V/(V_+ \oplus \mathfrak{sl}(2, \mathbf{C}) \oplus \mathbf{C}z^{-1}I)\}$$

of Gr_{-4} is an open subset, in fact, the open cell (or “big cell”) of a cell decomposition of Gr_{-4} . The foregoing lemma shows that W_0 is actually an element of this open subset:

$$(81) \quad W_0 \in \text{Gr}_{-4}^\circ.$$

The set

$$(82) \quad \mathcal{M} = \{W \in \text{Gr}_{-4}^\circ \mid W = W_0 \phi(z), \phi(z) \in G_{\text{in}}\}$$

of dressed vacua becomes the phase space of a dynamical system to which the Landau-Lifshitz hierarchy is mapped.

7.4 Interpretation of factorization problem

The following consideration is, again, limited to a small neighborhood of $t = 0$. In this situation, one can prove the following in the same way as the case of the nonlinear Schrödinger hierarchy.

Lemma 11 $W_0\chi(t, z) = W_0$.

Using this lemma, one can repeat the calculations done for the previous cases to show that the motion of the dressed vacuum $W(t) = W_0\phi(t, z) \in \mathcal{M}$ obeys the exponential law

$$(83) \quad W(t) = W(0) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right).$$

The converse, namely, deriving a solution of the factorization problem from the exponential flows needs an extra effort because the definition of the big cell is different from the previous cases. This is also related to the fact that the leading term $\phi_0(t)$ of $\phi(t, z)$ is generally not equal to I . A clue here is the fact that $W(t)$, as an element of the big cell, satisfies the condition that

$$(84) \quad \dim \operatorname{Im}(W(t) \rightarrow V/V_+) \cap \mathfrak{gl}(2, \mathbf{C}) = 1.$$

The leading term $\phi_0(t)$ is picked out from this one dimensional subspace; if t is sufficiently small, $\phi_0(t)$ is an invertible matrix. The rest of the construction is almost parallel to the case of the nonlinear Schrödinger hierarchy; see the paper [25] for details.

The conclusion is that the Grassmannian perspective also holds for this case, but with a different Grassmann variety:

Theorem 5 *The Landau-Lifshitz hierarchy can be mapped, by the correspondence $W(t) = W_0\phi(t, \lambda)$, to a dynamical system on the set \mathcal{M} of dressed vacua in the Grassmann variety Gr_{-4} . The motion of $W(t)$ obeys the exponential law. Conversely, the exponential flows on \mathcal{M} yield a solution of the factorization problem.*

8 Conclusion

A main conclusion of this case study is that the structure of a holomorphic vector bundle is the most important clue to the Grassmannian perspective of soliton equations with a zero-curvature representation constructed on an algebraic curve. This is also the case for classical soliton equations with a rational spectral parameter; the relevant holomorphic vector bundle therein is a trivial bundle. The two examples with an elliptic spectral parameters examined here are respectively accompanied by a bundle of its own particular type. The bundle for the elliptic nonlinear Schrödinger hierarchy is naturally the one in the

Tyurin parametrization. The bundle for the Landau-Lifshitz hierarchy is a rigid bundle.

It is remarkable that the mapping to an infinite dimensional Grassmann variety can be constructed in a fully parallel, almost universal way. Namely, the first thing to do is to choose a special base point W_0 , called “vacuum,” of the Grassmann variety. This is determined by the relevant holomorphic vector bundle E . More precisely, one has to choose a marked point P_0 of Γ , a local coordinate z in a neighborhood of P_0 and a local trivialization of E in a neighborhood of P_0 as extra geometric data. W_0 consists of Laurent series that represent (via the local trivialization of E) a holomorphic section of E over $\Gamma \setminus \{P_0\}$. The vacuum W_0 is then “dressed” by a Laurent series $\phi(z)$, which is related to changing the local trivialization of E . These geometric data are familiar stuff in the theories of algebro-geometric solutions of soliton equations, commutative rings of differential operators, etc. [9, 10, 11, 15, 16, 17, 18, 21].

This geometric point of view is already enough to tackle more general cases. It is rather straightforward to generalize the result for the elliptic analogue of the nonlinear Schrödinger hierarchy to higher genera, though explicit formulas of the A -matrices are not available therein. The work of Li and Mulase [17, 15], too, provides valuable material to this issue.

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